

Fermion dark matter from SO(10)

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Abstract

We construct and analyze nonsupersymmetric SO(10) standard model extensions which explain dark matter (DM) through the fermionic Higgs portal. In these SO(10)-based models the DM particle is naturally stable since a Z_2 discrete symmetry, the matter parity, is left at the end of the symmetry breaking chain to the standard model. Potentially realistic models contain the **10** and **45** fermionic representations from which a neutralino-like mass matrix with arbitrary mixings can be obtained. Two different SO(10) breaking chains will be analyzed in light of gauge coupling unification: the standard path $SU(5) \times U(1)_X$ and the left-right symmetry intermediate chain. The former opens the possibility of a split supersymmetric-like spectrum with an additional (inert) scalar doublet, while the later requires additional exotic scalar representations associated to the breaking of the left-right symmetry.

Keywords: Dark matter, Left-right symmetry, LHC, GUT

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I. INTRODUCTION

In view of the lack of signals of new physics in strong production at the LHC, the naturalness criterion as a guide to build extensions of the standard model has been losing priority in favor of other theoretical and phenomenological motivations.

Split supersymmetry (split-SUSY) [1–3] for example, gives up the explanation of the hierarchy problem while keeping the other main virtue of the minimal supersymmetric standard model: the connection between gauge coupling unification (GCU) and viable dark matter (DM) candidates without the imposition of any *ad hoc* discrete symmetry. In fact, the discrete symmetry required to avoid fast proton decay in supersymmetry can be embedded in an anomaly-free gauge symmetry (see for instance [4–8]) in order to avoid quantum gravitational effects which would violate it [9–11]. If in addition, the emerging discrete symmetry also forbids lepton (L) and baryon number (B) violation in the superpotential, the lightest supersymmetric particle is rendered stable with the potential to be a good dark matter candidate [12–14].

One straightforward possibility arises if split-SUSY is built in the framework of $SO(10)$ -GUT [15]. If we break the $U(1)_{B-L}$ subgroup of $SO(10)$ by the vacuum expectation values (VEVs) of fields with even $B - L$, then the discrete symmetry $P_M = (-1)^{3(B-L)}$, known as matter parity [16, 17], is preserved. In such a case, both the proton and dark matter stability are guaranteed at the renormalizable level.

It is interesting to stress that this possible DM stability explanation is independent of supersymmetry and can also happen when the Standard Model (SM) is embedded in $SO(10)$. [18]¹ Being a rank 5 group it contains an additional $U(1)_X$ subgroup, apart from the SM group, and if it is spontaneously broken by a scalar field S having a nonzero $U(1)_X$ charge X_S with $X_S = 0 \pmod{N}$ and $N \geq 2, 3, \dots$, then a remnant Z_N symmetry is expected to be present even at low energies. This makes $SO(10)$ to be a promising group to explain the origin of the DM stability [20]. Hence, some simplified DM models have been analyzed in light of the stability from $SO(10)$. In particular, the scalar doublet and singlet dark matter have been studied in [20, 21], the triplet (singlet) fermion DM was considered in

¹ In the minimal dark matter scenario [19], the DM candidate is either a scalar septuplet or fermion quintuplet of $SU(2)_L$ and its stability is guaranteed by the SM gauge symmetry.

Refs. [22, 23] ([24]), whereas the radiative seesaw model was analyzed in Ref. [25].

In the last reference, it was also shown that a robust GCU can be obtained in $\text{SO}(10)$ when the set of low energy fields emerging from the even $B - L$ fermionic representations $\mathbf{10}_F$ and $\mathbf{45}_F$ matches the particle content of split-SUSY with one additional scalar doublet. In this way, the spectrum matches exactly the low energy particle content of partial split-SUSY (PSS) [26]. To our knowledge, this minimum set of fields was first proposed in Ref. [27].

In this paper, we show that with the Yukawa couplings of the low energy fields associated to the mixing of $\mathbf{10}_F$ and $\mathbf{45}_F$, through the Higgs field in $\mathbf{10}_H$, we can obtain a neutralino-like mass matrix but with different mixings compared to the usual gauginos and Higgsinos. In this way, our framework automatically gives an explanation for the origin of the *ad hoc* discrete symmetry of simplified fermion dark matter models connected to the Higgs portal [28]. In particular, we formulate $\text{SO}(10)$ realizations of the singlet-doublet fermion dark matter (SDFDM) model [29–32] and the doublet-triplet fermion dark matter (DTFDM) model [33].

Scalar and vector DM naturally make use of the Higgs portal through the invariant Higgs mass factor $H^\dagger H$ whereas fermion DM requires an ultraviolet (UV) realization, via a scalar or a vector mediator, of the dimension-5 terms $\bar{F} F H^\dagger H$ and $\bar{F} \gamma_5 F H^\dagger H$ [34]. The singlet fermion DM model [35–37] is a UV realization of the fermionic Higgs portal [28, 38–40] with an additional singlet scalar (to be mixed with the Higgs) as the mediator. On the other hand, for those simplified models where the DM particle is a mixture of either singlet and doublet fermions or doublet and triplet fermions, the mediator particle is the Higgs itself. Thus the SDFDM and DTFDM models are two of the simplest fermionic DM models where the Higgs portal is open without additional scalar degrees of freedom. This, along with the Higgs boson discovery and the lack of signals of new physics in strong production at the LHC, make of these simplified fermion DM models (where the production of new particles is only through electroweak processes) a realistic and promising solution to the DM puzzle.

With the PSS-like spectrum as in [25], we revisit the GCU with emphasis in a scenario where $\text{SO}(10)$ breaks to the SM through the $\text{SU}(5) \times \text{U}(1)_X$ chain.

Finally, we explore the possibility to have a correct non-SUSY $\text{SO}(10)$ GCU with another kind of spectrum in the ballpark of the $O(1)$ TeV. Since the triplet fermion DM model only

requires one additional colored octet at some high scale to achieve GCU [22, 23], we will focus in the case when only the singlet-doublet fermions contains the DM candidate. There exists much literature which already discusses GCU for the breaking chain of $\text{SO}(10)$ containing the left-right (LR) symmetric gauge group with remnant gauge $U(1)_{B-L}$ symmetry [30, 41–46]. We will check specifically if SDFDM is compatible with a low LR intermediate symmetry breaking, adding at this level a few extra particle content imposed to pass some specific phenomenological constraints.

In the next section, we present the minimal $\text{SO}(10)$ setup to realize the fermionic DM Higgs portal. In the Sec. III, we analyze the GCU for some models which successfully constitute a fermionic DM Higgs portal realization, and interesting configurations will be explored. Finally, in Sec. IV we present our conclusions.

II. FERMION DM FROM $\text{SO}(10)$

The split-SUSY scenario demands that the supersymmetric partners of the leptons and quarks along the second Higgs doublet stay at an intermediate scale $M_S \gg 1$ TeV, whereas the first Higgs doublet, Higgsinos and gauginos remain at low energies [1–3]. Therefore, any nonsupersymmetric version of such a scenario involves the following particle content: a hyperchargeless singlet Weyl fermion N , two $\text{SU}(2)_L$ -doublets Weyl fermions χ, χ^c with opposite hypercharge $Y = \pm 1/2$, a hyperchargeless $\text{SU}(2)$ -triplet Weyl fermion Σ and a color octet Weyl fermions Λ with $Y = 0$. To generate this particle spectrum from $\text{SO}(10)$, we choose its P_M -even vector $\mathbf{10}_F$ and adjoint $\mathbf{45}_F$ fermion representations [25]. Concretely, N, Σ and Λ belong to the adjoint representation, and the Weyl doublets are in the vectorial one. As usual the SM fermions are in the P_M -odd spinorial $\mathbf{16}_a$ representation ($a = 1, 2, 3$ is the family index), while the Higgs field is assigned to the fundamental representation $\mathbf{10}_H$. In this way, the matter parity guarantees the stability of the dark matter particle, which is a mixture of all the P_M -even neutral colorless fermions in the spectrum.

The most general $\text{SO}(10)$ invariant Lagrangian contains the following Yukawa terms

$$-\mathcal{L} \supset Y \mathbf{10}_F \mathbf{45}_F \mathbf{10}_H + M_{\mathbf{45}_F} \mathbf{45}_F \mathbf{45}_F + M_{\mathbf{10}_F} \mathbf{10}_F \mathbf{10}_F. \quad (1)$$

To break the mass degeneracy within $\mathbf{10}_F$ and $\mathbf{45}_F$ multiplets and at the same time generate low scale masses for the nonstandard fermions it is enough to consider the additional scalar

representations $\mathbf{45}_H$, $\mathbf{54}_H$ and $\mathbf{210}_H$ [23, 25]. Concretely, the Lagrangian involving the $\mathbf{10}_F$ and $\mathbf{45}_F$ mass terms reads [23, 25, 47]

$$\mathcal{L}_{\mathbf{10}_F+\mathbf{45}_F}^{\text{mass}} = \mathbf{10}_F (M_{\mathbf{10}_F} + h'_e \langle \mathbf{54}_H \rangle) \mathbf{10}_F + \mathbf{45}_F (M_{\mathbf{45}_F} + h_e \langle \mathbf{54}_H \rangle + h_p \langle \mathbf{210}_H \rangle) \mathbf{45}_F. \quad (2)$$

Since $\mathbf{210}_H$ have three singlets, while $\mathbf{54}_H$ has only one, the full set of masses are

$$\begin{aligned} m(1, 2, 1/2) &= M_{\mathbf{10}_F} + \frac{3h'_e}{2} \langle \mathbf{54}_H \rangle, \\ m(3, 1, -1/3) &= M_{\mathbf{10}_F} - h'_e \langle \mathbf{54}_H \rangle, \\ m(3, 1, 2/3) &= M_{\mathbf{45}_F} + \sqrt{2}h_p \frac{\langle \mathbf{210}_H \rangle_2}{3} - 2h_e \frac{\langle \mathbf{54}_H \rangle}{\sqrt{15}}, \\ m(3, 2, 1/6) &= M_{\mathbf{45}_F} + h_p \frac{\langle \mathbf{210}_H \rangle_3}{3} + h_e \frac{\langle \mathbf{54}_H \rangle}{2\sqrt{15}}, \\ m(3, 2, -5/6) &= M_{\mathbf{45}_F} - h_p \frac{\langle \mathbf{210}_H \rangle_3}{3} + h_e \frac{\langle \mathbf{54}_H \rangle}{2\sqrt{15}}, \\ m(1, 1, 0) &= m(1, 1, 1) = M_{\mathbf{45}_F} + \sqrt{2/3}h_p \langle \mathbf{210}_H \rangle_1 + \sqrt{3/5}h_e \langle \mathbf{54}_H \rangle, \\ m'(1, 1, 0) &= M_{\mathbf{45}_F} + \frac{2\sqrt{2}}{3}h_p \langle \mathbf{210}_H \rangle_2 - \frac{2}{\sqrt{15}}h_e \langle \mathbf{54}_H \rangle, \\ m(8, 1, 0) &= M_{\mathbf{45}_F} - \frac{\sqrt{2}}{3}h_p \langle \mathbf{210}_H \rangle_2 - \frac{2}{\sqrt{15}}h_e \langle \mathbf{54}_H \rangle, \\ m(1, 3, 0) &= M_{\mathbf{45}_F} - \sqrt{\frac{2}{3}}h_p \langle \mathbf{210}_H \rangle_1 + \sqrt{\frac{3}{5}}h_e \langle \mathbf{54}_H \rangle. \end{aligned}$$

Solving in terms of $M_D = m(1, 2, 1/2)$, $M_\Lambda = m(8, 1, 0)$, $M_\Sigma = m(1, 3, 0)$, and $M_N = m'(1, 1, 0)$, we have that all the other masses are of order $M_{\mathbf{10}_F}$, $M_{\mathbf{45}_F} \sim m_G$, except for

$$M_T = m(3, 1, 2/3) = (M_\Lambda + 2M_N) / 3. \quad (3)$$

Therefore the fermion spectrum (FS) at low-intermediate energies can involve N, χ, Σ, Λ and/or T . Namely we have the following possibilities for the fields belonging to $\mathbf{45}_F$ having arbitrary masses, i.e., their masses are free parameters:

$$\text{FS}_{\mathbf{45}_F} \text{ I : } \quad \Sigma, \Lambda, \quad \text{with} \quad M_N, M_T \sim M_G, \quad (4)$$

$$\text{FS}_{\mathbf{45}_F} \text{ II : } \quad N, \Sigma, \quad \text{with} \quad M_\Lambda, M_T \sim M_G, \quad (5)$$

$$\text{FS}_{\mathbf{45}_F} \text{ III : } \quad N, \Lambda, \Sigma, T. \quad (6)$$

It is worth mentioning that for the fermion spectrum (6) the VL up-type quark T is required due to Eq. (3). However, if a second innocuous $\mathbf{45}_F$ is introduced with the corresponding

singlet $m'(1, 1, 0)$ having an arbitrary mass such a VL quark can be removed from the spectrum (6) ($M_N, M_T \sim m_G$), leading to another spectrum comprising the spectrum (4) plus a new singlet denoted again as N :

$$\text{FS}_{45_F} \text{ IV : } \quad N, \Lambda, \Sigma. \quad (7)$$

The SO(10) breaking leads to the effective DM Yukawa Lagrangian for the fermion spectrum (6) with the pair χ, χ^c ,²

$$\begin{aligned} \mathcal{L}_{eff} = & M_D \chi^c \chi - \frac{1}{2} M_N N N - \frac{1}{2} M_\Sigma \Sigma \Sigma \\ & - y_1 H \chi^c N - y_2 \tilde{H} \chi N + f_1 H \epsilon \Sigma \chi^c - f_2 \tilde{H} \epsilon \Sigma \chi + \text{h.c.} \end{aligned} \quad (8)$$

In this way, the opening of the Higgs portal through the y_i and f_i terms allows the construction of the general scenario of singlet-doublet-triplet fermion DM, a neutralino-like scenario. After the electroweak symmetry breaking the y_i and f_i terms induce a mixture between all the colorless P_M -even neutral fermions, and thus also a breaking of the mass degeneracy between the neutral parts of the doublet fermions χ and χ^c . It follows that the particle spectrum consists of four Majorana fermions and two Dirac charged fermions. The neutral fermion mass matrix in the basis $\psi^0 = (N, \Sigma^0, \chi^{c0}, \chi^0)^T$ reads

$$\mathcal{M}_{\psi^0} = \begin{pmatrix} M_N & 0 & -y \cos \beta v / \sqrt{2} & y \sin \beta v / \sqrt{2} \\ 0 & M_\Sigma & f \cos \beta' v / \sqrt{2} & -f \sin \beta' v / \sqrt{2} \\ -y \cos \beta v / \sqrt{2} & f \cos \beta' v / \sqrt{2} & 0 & -M_D \\ y \sin \beta v / \sqrt{2} & -f \sin \beta' v / \sqrt{2} & -M_D & 0 \end{pmatrix}, \quad (9)$$

while charged fermion mass matrix in the basis $\psi^+ = (\Sigma^+, \chi^+)^T$ and $\psi^- = (\Sigma^-, \chi^{c-})^T$ is given by

$$\mathcal{M}_{\psi^\pm} = \begin{pmatrix} M_\Sigma & f \sin \beta' v \\ f \cos \beta' v & M_D \end{pmatrix}. \quad (10)$$

² Here we use the additional scalar representations $\mathbf{120}_H$ and $\mathbf{320}_H$, along the renormalization group equations, to generate a hierarchy between the four Yukawa couplings y_i and f_i at low energies. This implies that the Higgs doublet H is a linear combination of the weak doublets present in $\mathbf{10}_H$, $\mathbf{120}_H$ and $\mathbf{320}_H$. Another way to generate such a hierarchy is taking the $\mathbf{10}_H$ as complex [48]. In that case, to avoid the coupling of the SM fermions to $\mathbf{10}_H^*$ an additional global $U(1)_{PQ}$ symmetry may be imposed leading to the axion as the DM candidate [48, 49]. Because we are interested in WIMP fermion DM, we will not consider this case here.

Here $y = \sqrt{y_1^2 + y_2^2}$, $f = \sqrt{f_1^2 + f_2^2}$, $\tan \beta = y_2/y_1$ and $\tan \beta' = f_2/f_1$. These mass matrices have the typical structure of the very well-known neutralino and chargino mass matrices in the minimal supersymmetric standard model (MSSM) [50]. Indeed, the supersymmetric case corresponds to the limit $y = g'/\sqrt{2}$, $f = g/\sqrt{2}$ and $\tan \beta = \tan \beta'$.

It is worth mentioning the crucial role of the mixing terms in the neutral fermion sector. In the absence of them, the singlet DM would not couple to the SM particles thus leading a large relic abundance while the doublet DM would be excluded due to the coupling to the Z gauge boson which gives rise to a spin-independent cross section orders of magnitude larger than present limits. The only limiting case that does not require the mixing terms is the triplet DM one.

The present fermion particle spectrum was considered in Ref. [51] with the aim of strengthen the first order electroweak phase transition in order to have a successful electroweak baryogenesis. A neutralino-like mass matrix was also realized in fake split-SUSY [52] but with suppressed mixings between fake gauginos and fake higgsinos.

On the other hand, it is also possible to generate simpler DM scenarios by assuming a mass hierarchy among the neutral fermions. For $M_\Sigma \gg M_N, M_D$ the simplified model of SDFDM [29–32] is obtained³, whereas the DTFDM model [33, 40, 53] emerges when $M_N \gg M_\Sigma, M_D$. Of course, the triplet fermion DM model [19] is also possible as long as $M_\Sigma \ll M_N, M_D$ [22, 23].

The phenomenology of the model in direct and indirect dark matter detection experiments, and in colliders, is usually studied in the limits of simplified fermion dark matter through the Higgs portal [40, 53], with emphasis in couplings which depart from SUSY limit. In this way, the SDFDM has been thoroughly studied in several works [29–32, 40, 53–58]. The dark matter candidate is the lightest state coming from the mixing of the neutral component of the doublet and the neutral singlet. When the dark matter candidate is mainly singlet (doublet) the relic density is in general rather large (small). In particular, a pure doublet has the proper relic density for $M_\chi \sim 1$ TeV [30, 55, 59]. The LHC phenomenology was analyzed in [53]. Their conclusion, is that the recast of the current LHC data is easier to evade, but the long-rung prospects are promising, since the region $M_N, y_1 v, y_2 v \ll M_D$

³ One alternative GUT scenario to have singlet-doublet fermion DM was presented in [30].

could be probed up to $M_D \lesssim 600 - 700$ GeV for the 14-TeV run of the LHC with 3000 fb^{-1} .

On the other hand, the phenomenology of the DTFDM has been studied in [33, 40, 53]. The dark matter candidate is the lightest state coming from the mixing of the neutral components of the doublet and the triplet. In the low DM mass region, the relic density is properly satisfied in the range $0 \leq (M_D, M_\Sigma) \leq 400$ GeV and $0 \leq (f, f') \leq 1.5$. However this region is excluded due to the contribution of the new charged fermions to the Higgs diphoton decay [53]. For the high DM mass region, the expectations are analogous to the ones of the doublet or triplet fermion DM, where a large value for the DM mass is required. When the doublet is decoupled, the triplet fermion dark matter model is recovered with a mass of ~ 2.7 GeV to explain the correct relic abundance [19]. Therefore, its phenomenology at near-future colliders is quite limited [53].

III. SO(10) UNIFICATION

As it is well known, in non-SUSY SO(10) scenarios, the unification of the gauge couplings can be as good as, or even better than, in the MSSM, despite that the number of extra fields up to the SM is small. This extra particle content can successfully fulfill all the constraints coming from the fermion masses, proton decay, and perturbativity. In addition, if more restrictive conditions are imposed like a simplified DM model spectrum, the required extra field content needs to be more specific. In what follows, we will concentrate on these kinds of non-SUSY SO(10) scenarios, focusing on two different channels to break SO(10) to the SM, containing each one the remnant $U(1)_{B-L}$ symmetry necessary to stabilize DM. The first scenario to analyze is based on the $\text{SO}(10) \rightarrow \text{SU}(5) \times \text{U}(1)_X$ breaking channel. Here, a PSS-like spectrum with singlet-doublet-triplet fermion DM is considered. One well-known possibility in this chain is to have triplet fermion dark matter at low energy with a fermion octet at one intermediate scale. In order to have only singlet-doublet dark matter at low energies, we explore a second scenario based in the left-right symmetry breaking chain. Very simple configurations of fields which not only explain rich phenomenology but also DM through the singlet-doublet DM realization are analyzed.

A. Partial split supersymmetry-like model

Here, we consider the symmetry breaking channel:

$$\text{SO}(10) \rightarrow \text{SU}(5) \times \text{U}(1)_X \rightarrow \text{SM}. \quad (11)$$

In order to avoid intermediate breaking scales, we assume that $\text{SU}(5) \times \text{U}(1)_X$ breaks to the SM also at the unification scale m_G , joint with the $\text{SO}(10)$ symmetry breaking. At the first step, we are adding the two fermion doublets χ, χ^c at the electroweak scale $m_{\text{EW}} = 100$ GeV. At this scale an extra configuration of fields, denoted as X , is added such that $\text{SM} + \chi + \chi^c + X$ unifies equal or better than the MSSM at a scale of m_G . The X configuration and the unification scale m_G depend on the value of the new physics scale m_{NP} . As a first example, if we assume $m_{\text{NP}} = m_{\text{EW}}$, that is, the doublet fermions and the rest of fields are added at the electroweak scale, one of the simplest and interesting configurations found corresponds to $X = \Phi_{1,2,1/2} + 2\Phi_{1,3,0} + 2\Phi_{8,1,0}$ [25] which unifies at a scale of $m_G = 2 \times 10^{16}$ GeV, when $M_D = 100$ GeV. A more general scan is to be presented below. This configuration is also denoted in the literature as $\Phi_{1,2,1/2} + \Psi_{1,3,0} + \Psi_{8,1,0}$ since in our case two scalar fields, 2Φ , correspond to one fermionic field Ψ . Note that $X = \Phi_{1,2,1/2} + \Psi_{1,3,0} + \Psi_{8,1,0} + N$ (with two 45_F , see Eq. (7)) is the same split-SUSY configuration but with the second scalar doublet ($\Phi_{1,2,1/2}$) living at low energies, which has been dubbed as partial split-SUSY [26]. Furthermore, $X + \chi + \chi^c$ has the same fermion fields of the singlet-doublet-triplet fermion DM scenario discussed in the previous section. Therefore, that $\text{SO}(10)$ -based scenario is compatible with gauge coupling unification. Of course, the DTFDM model is also compatible with GCU since the SM singlets have no impact on it. With regard to the additional scalar doublet, if it proceeded from the $\mathbf{16}_H$ representation, which is P_M -odd, and did not develop a VEV, then the dark matter stability would be still guaranteed by the matter parity as first noted in [20, 21]⁴. Moreover, with the $\Psi_{1,3,0}$ and N from the two 45_F and $\Phi_{1,2,1/2}$ from the $\mathbf{16}_H$, it is possible to build a hybrid type-III [63] and type-I [64] radiative seesaw in $\text{SO}(10)$ as analyzed in [25].

However, there is a more economical possibility by using a single 45_F evolving the spectrum (6). In that case we need to consider the effect of the T color triplet in the running

⁴ In other words, the second scalar doublet would be an inert doublet [60–62].

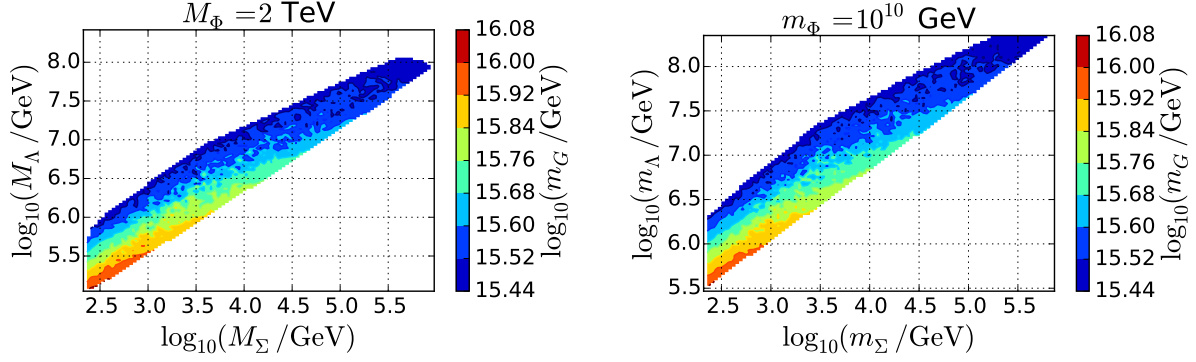


Figure 1: $SU(2)$ -triplet fermion mass as a function of the $SU(3)$ -octet fermion mass for $M_\Phi = 2$ TeV (left panel), and $M_\Phi = 10^{10}$ GeV (right panel). The red (blue) colors in the lower (upper) part of the region signal for high (low) unification scales compatible with proton decays.

of the renormalization group equations (RGEs). To study the possibility of unification scale in that case, we scan the parameter space with

$$0 \leq M_N/\text{GeV} \leq 3000, \quad 100 \leq M_D/\text{GeV} \leq 3000, \quad (12)$$

with either $M_\Sigma > \min(M_N, M_D)$ or $M_\Sigma = 2.7$ TeV. The lightest neutral eigenvalue from the mass matrix in Eq. (9) is checked to have the proper dark matter relic density and compatibility with all the phenomenological constraints as explained in Ref. [57]. For each point in the scan we check if it is possible to choose M_Λ and M_Φ to get proper unification within the range $3 \times 10^{15} < m_G/\text{GeV} < 10^{18}$. The results are shown in Fig. 1. There we show M_Λ as a function of M_Σ for $M_\Phi = m_{\Phi(1,2,1/2)} = 2$ TeV (left panel) and $M_\Phi = 10^{10}$ GeV (right panel). In both figures, the several colors represent the possible values of m_G , ranging from the dark-blue color for $m_G \approx 3 \times 10^{15}$ GeV to the red color for $m_G \approx 1.2 \times 10^{16}$ GeV. In this way, large unification scales are obtained for low M_Σ and M_Λ , with a minimum value of M_Λ around 100 TeV for $M_\Phi = 2$ TeV and 300 TeV for $M_\Phi = 10^{10}$ GeV. We can see that the effect of the doublet scalar is to rescale the mass of the color octet with a factor of 3 for their quoted values. Moreover, the results have only a mild dependence on the specific choice of M_D and M_N when the RGEs at one-loop are used to analyze unification.

For completeness, we also show the lower M_Λ scale allowed in the case of two $\mathbf{45}_F$ in Fig. 2 when $M_\Phi = 2$ TeV. We can see that the unification scale at $m_G = 2 \times 10^{16}$ GeV

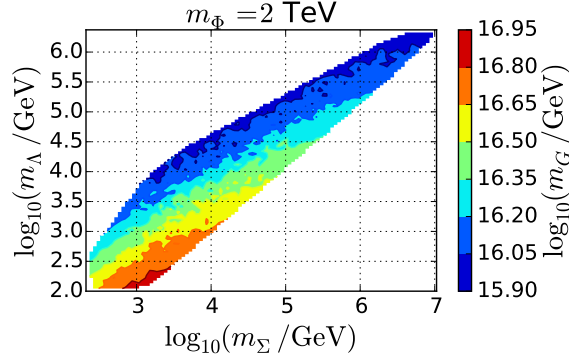


Figure 2: The same as in left panel of Fig. 1, but with two $\mathbf{45}_F$

can be obtained from close to electroweak scale values for M_Σ (M_Λ), until $M_\Sigma \lesssim 1000$ TeV ($M_\Lambda \lesssim 100$ TeV).

The required colored octet has been shown in [22, 25], to have an abundance and lifetime sufficiently small to satisfy all experimental constraints [1]. In particular, if the lifetime of the octet is long enough, this would hadronize into R-hadrons as in split-SUSY, and the limits from ATLAS [65] or CMS [66] for this kind of states would apply. In the later experimental study, a colored octet with mass less than 880 GeV is excluded if it decays into a gluon and the DM fermion candidate with a branching of 100% and a lifetime between $1 \mu\text{s}$ and 1000 s, providing that the energy of the final gluon is larger than 120 GeV.

The fermion spectrum given (4) was used in Ref. [67] where a DM scheme arise from a simple unification configuration containing only a fermion DM triplet at low energies at the price of having a fermion octet at high energies [67] (the Dirac fermion case is analyzed in [68]). To have the proper DM relic abundance with a fermion-triplet of 2.7 TeV, the fermion octet needs to have a mass in a narrow range around 2×10^{10} GeV.

We now check if it is possible to have a pure SDFDM realization with an intermediate left-right symmetry scale.

B. Breaking through left-right chain

As previously motivated, this scenario represents another possibility to link the DM with GCU. In this case, we will concentrate only in the singlet-doublet fermion DM scenario,

Field	Multiplicity	$3_c 2_L 2_R 1_{B-L}$	Spin	SO(10) origin
Q	3	$(3, 2, 1, +\frac{1}{3})$	1/2	16
Q^c	3	$(\bar{3}, 1, 2, -\frac{1}{3})$	1/2	16
L	3	$(1, 2, 1, -1)$	1/2	16
L^c	3	$(1, 1, 2, +1)$	1/2	16
Φ	1	$(1, 2, 2, 0)$	0	10
χ, χ^c	1	$(1, 2, 2, 0)$	1/2	10
N	1	$(1, 1, 1, 0)$	1/2	45

Table I: The relevant part of the field content. Note that, the two fermion doublets χ and χ^c come from an only fermionic LR bidoublet. In the third column the relevant fields are characterized by their $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ quantum numbers while their SO(10) origin is specified in the fourth column.

showing some simple LR configuration of fields which fulfill some other phenomenological requirements.

For this model construction, we consider a chain in which SO(10) is broken in exactly one intermediate LR step to the standard model group as:

$$SO(10) \rightarrow SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \rightarrow SM. \quad (13)$$

The left-right symmetry breaking scale (denoted in this case as m_{LR}) can be as low as $O(1)$ TeV or as high as, said 10^9 GeV, maintaining nevertheless gauge coupling unification into the scheme of SO(10). In Refs. [45, 46, 69] SO(10) models with an intermediate left-right symmetry have been studied. There, simple configurations which unify and also contain some interesting phenomenological aspects have been explored.

For the construction of our configurations, we have taken the basic particle content described in Table I which consists of the SM fermions, plus the SM Higgs (Φ), and the new particle content of the SDFDM.

Therefore, at this point, the β -function contributions of the basic fields in Table I for

the two regimes $[m_{\text{EW}}, m_{\text{LR}}]$ and $[m_{\text{LR}}, m_G]$ are given as:

$$\begin{aligned} (b_3^{\text{SM}}, b_2^{\text{SM}}, b_1^{\text{SM}}) &= (-7 + \Delta b_3^{DM}, -19/6 + \Delta b_2^{DM}, 41/10 + \Delta b_1^{DM}), \\ (b_3^{\text{LR}}, b_2^{\text{LR}}, b_R^{\text{LR}}, b_{B-L}^{\text{LR}}) &= (-7, -7/3, -7/3, 4) + (\Delta b_3^{\text{LR}}, \Delta b_2^{\text{LR}}, \Delta b_R^{\text{LR}}, \Delta b_{B-L}^{\text{LR}}), \end{aligned} \quad (14)$$

where the $\Delta b_i^{DM} = (0, 2/3, 2/5)$ are the contributions of the two additional fermion DM doublets χ and χ^c . We are using the canonical (C) normalization for the $(B - L)$ charge related to the physical (P) one, by $(B - L)^C = \sqrt{3/8}(B - L)^P$. Here, the Δb_i^{LR} stands for the contribution of the additional fields which are added at the LR intermediate scale. It is clear that after adding only two fermionic doublets to the SM particle content, the gauge couplings still do not unify. Only once the additional fields are added at the LR regime, unification is achieved at a scale of about $[10^{15}, 10^{17}]$ GeV (fulfilling this the actual bounds that proton decay imposes in the GUT scale). As previously mentioned, χ and χ^c are added at the SM scale, so the interactions of DM with the LR gauge bosons and the other particles in this regime do not arise in this scenario. To construct our models, besides to impose the presence of χ , and χ^c and N at the SM level, we require also a number of additional conditions for a model to be both realistic and phenomenological interesting: (i) all models must have the agents to break the LR symmetry to the SM group (this achieved by the field $\Phi_{1,1,3,-2}$), (ii) all models must contain (at least) one of the minimal ingredients to generate a realistic CKM in the quark sector, i.e, at least one copy of the $\Phi_{1,2,2,0}$ bidoublet and a copy of the $\Phi_{1,1,3,0}$ right triplet, iii) models must have perturbative gauge couplings, and (iv) m_G should be large enough to prevent too rapid proton decay, i.e, $m_G \geq 3 \times 10^{15}$ GeV [70].

Note that m_{LR} should be low enough so that the new fields can be accessible at the experiment, similar to the analysis already done in [69]. However, for completeness, we will show the simplest configurations of field even for large values of m_{LR} as is depicted in Table II.

The simplest of all the benchmark models passing the unification conditions above mentioned, with a left-right scale significantly low ($m_{\text{LR}} = 2$ TeV) is $\Phi_{1,1,3,0} + \Phi_{8,1,1,0} + 2\Phi_{1,1,3,-2}$. Figure 3 shows the running of the gauge couplings for this simple model. Note that all the fields in the LR regime are added at the LR scale of 2 TeV. Considering this relatively low value of mass for the octet, there should be a chance for $\Phi_{8,1,0}$ to be within the reach of the current run of the LHC. The study of this scalar octet production has been already covered

m_{LR} (GeV)	LR configuration	m_G (GeV)
2×10^3	$\Phi_{1,1,3,0} + \Phi_{8,1,1,0} + 2\Phi_{1,1,3,-2}$	2.47×10^{17}
	$3\Phi_{1,1,3,0} + \Phi_{1,2,2,0} + \Phi_{8,1,1,0} + 2\Phi_{1,1,3,-2}$	1.65×10^{16}
	$\Phi_{1,1,3,0} + 3\Phi_{3,1,1,4/3} + 2\Phi_{1,1,3,-2}$	5.02×10^{15}
10^5	$\Phi_{1,1,3,0} + \Phi_{8,1,1,0} + 2\Phi_{1,1,3,-2}$	1.01×10^{17}
	$2\Phi_{1,1,3,0} + \Phi_{1,2,2,0} + \Phi_{8,1,1,0} + 2\Phi_{1,1,3,-2}$	1.01×10^{16}
	$\Phi_{1,1,3,0} + 3\Phi_{3,1,1,4/3} + 2\Phi_{1,1,3,-2}$	3.67×10^{15}
10^7	$\Phi_{1,1,3,0} + \Phi_{8,1,1,0} + 2\Phi_{1,1,3,-2}$	3.55×10^{16}
	$\Phi_{1,1,3,0} + \Phi_{1,2,2,0} + \Phi_{8,1,1,0} + 2\Phi_{1,1,3,-2}$	5.69×10^{15}
	$\Phi_{1,1,3,0} + 2\Phi_{3,1,1,-2/3} + 3\Phi_{3,1,1,4/3} + \Phi_{1,1,3,-2}$	5.69×10^{15}

Table II: Simple LR configurations passing the constraints explained in the text. One of the scalar bidoublets Φ is already considered in the basic field content ($SM + \Phi + \chi + \chi^c$) as shown in Table I. The first configuration corresponds to the minimal solution. Both scales m_{LR} and m_G are given in GeV.

in the literature [71–73]. At the LHC one of the contribution comes from the gluon-gluon annihilation into two scalar octets $gg \rightarrow \Phi_{8,1,0}$. For light scalar octets two gluon and quark-antiquark annihilations into two scalar octets give an additional contribution to the two-jet cross section [71]. There are also effects of the scalar octet on the process $pp \rightarrow 4\text{jets}$ at the LHC, being this one of the best signatures to look for [72]. This is then an interesting solution that not only explains DM but also allows one to explore rich phenomenology coming from the colored octet at the LHC.

It is worth to stressing that a sufficiently low LR scale is still compatible with the interpretation for the ATLAS diboson excess [74] as a possible W_R resonance [75]. This will be reconsidered in future works where a detailed analysis of the interaction in the LR regime, in particular the $W_L - W_R$ possible mixings would be done.

It is interesting to note that in the standard left-right symmetric models (without considering any extra particle contribution in the regime $[m_{EW}, m_{LR}]$), the extra degree of freedom of having one intermediate m_{LR} scale, allows one to achieve gauge coupling unification even if the extra particle content does not contain colored fields. However in our models, before reaching the LR intermediate stage, there is a previous Δb_{2L} contribution

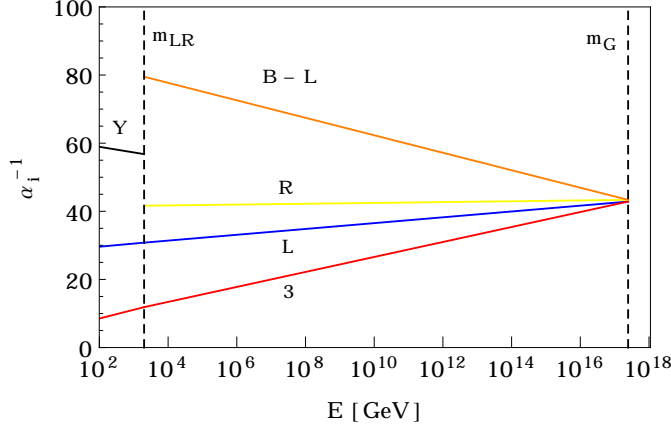


Figure 3: Running of the gauge couplings for the first simple configuration shown in Table II. In the regime $[m_{EW}, m_{LR}]$ live the fields $SM + \Phi_{1,2,1/2} + \chi_{1,2,1/2} + \chi_{1,2,-1/2}^c + N_{1,1,0}$. In the second $[m_{LR}, m_G]$ regime contribute the basic field depicted in Table I plus the extra fields: $\Phi_{1,1,3,0} + \Phi_{8,1,1,0} + 2\Phi_{1,1,3,-2}$.

which comes from the fermion doublets χ and χ^c already added at the SM scale, then some colored fields must be added at the LR scale in order to compensate this amount. On the other hand, if the scale at which the fermionic DM candidates are added is greater than m_{EW} and very close to m_{LR} it could be possible to obtain GCU without colored fields in the LR regime, but, all these solutions are excluded since the unification scale is very low, i.e, $m_G < 3 \times 10^{15}$ GeV.

IV. CONCLUSIONS

In this paper we have taken advantage of the fact that a Z_2 symmetry appears as a remnant symmetry at the end of the symmetry breaking chain of the $SO(10)$ GUT group to the standard model, for constructing simplified fermion dark matter models where the dark matter stability is naturally guaranteed.

Concretely, we have formulated a viable $SO(10)$ model capable of realizing at low energies the singlet-doublet-triplet fermion dark matter. The model engages as nonstandard fermions a SM singlet and a hyperchargeless weak triplet, both belonging to the $\mathbf{45}_F$, and a couple of weak doublets with $Y = \pm 1/2$ belonging to the $\mathbf{10}_F$ representation. The mixing between these fermions is carried out by the SM Higgs, which is assigned to the $\mathbf{10}_H$ rep-

resentation. At low energies the resulting particle spectrum resembles the neutralino and chargino sets of the MSSM but with the difference that the mixing terms are not controlled by the gauge couplings. Thanks to the versatility of the model, it is also possible to realize the simpler fermion DM scenarios of singlet-doublet, doublet-triplet or only triplet.

Regarding gauge coupling unification, we have shown that the model has a successful $SO(10)$ unification through the $SU(5) \times U(1)_X$ chain by requiring the additional presence of a scalar weak doublet and a fermion color octet. The DTFDM model shares this same feature while the SDFDM model requires the presence of other DM fields. However, this model under the left-right symmetry intermediate chain successfully achieves $SO(10)$ unification by demanding only the exotic scalar representations associated to the breaking of the left-right symmetry.

In summary, an interesting configuration of fields which pass some physical conditions such as a GCU, proton stability, compatibility with the quark and lepton masses and mixings, fermionic DM realization and some other nontrivial LHC phenomenology were found for the two $SO(10)$ breaking channels explored. For both cases, the new extra particle content close to the TeV scale, would make the new physics testable at the LHC test.

Note added: Recently, we became aware of the work of N. Nagata, K.A.Olive and J. Zheng [76] where singlet-doublet fermion dark matter model is studied in the limit where the singlet is at some high intermediate scale into the framework of $SO(10)$ unification.

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